

# Novel PT-invariant Kink and Pulse Solutions For a Large Number of Real Nonlinear Equations

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## Abstract:

For a large number of real nonlinear equations, either continuous or discrete, integrable or nonintegrable, uncoupled or coupled, we show that whenever a real nonlinear equation admits kink solutions in terms of  $\tanh \beta x$ , where  $\beta$  is the inverse of the kink width, it also admits solutions in terms of the PT-invariant combinations  $\tanh 2\beta x \pm i \operatorname{sech} 2\beta x$ , i.e. the kink width is reduced by half to that of the real kink solution. We show that both the kink and the PT-invariant kink are linearly stable and obtain expressions for the zero mode in the case of several PT-invariant kink solutions. Further, for a number of real nonlinear equations we show that whenever a nonlinear equation admits periodic kink solutions in terms of  $\operatorname{sn}(x, m)$ , it also admits periodic solutions in terms of the PT-invariant combinations  $\operatorname{sn}(x, m) \pm i \operatorname{cn}(x, m)$  as well as  $\operatorname{sn}(x, m) \pm i \operatorname{dn}(x, m)$ . Finally, for coupled equations we show that one cannot only have complex PT-invariant solutions with PT eigenvalue  $+1$  or  $-1$  in both the fields but one can also have solutions with PT eigenvalue  $+1$  in one field and  $-1$  in the other field.

# 1 Introduction

Nonlinear equations are playing an increasingly important role in several areas of science in general and physics in particular [1]. One of the major problems with these equations is the lack of a superposition principle. In particular, even if one can find two solutions, say  $\phi_1$  and  $\phi_2$ , of a given nonlinear equation, unlike the linear case, any linear combination of  $\phi_1$  and  $\phi_2$  is usually not a solution of that nonlinear equation. Thus if we can find some general results about the existence of solutions to nonlinear equations, that would be invaluable. In this context it is worth recalling that some time ago we [2, 3] have shown (through a number of examples) that if a nonlinear equation admits periodic solutions in terms of Jacobi elliptic functions  $\text{dn}(x, m)$  and  $\text{cn}(x, m)$ , then it will also admit solutions in terms of  $\text{dn}(x, m) \pm \sqrt{m} \text{cn}(x, m)$  where  $m$  is the modulus of the elliptic function [4]. Further, in the same papers [2, 3], we also showed (again through several examples) that if a nonlinear equation admits solutions in terms of  $\text{dn}^2(x, m)$ , then it will also admit solutions in terms of  $\text{dn}^2(x, m) \pm \text{cn}(x, m)\text{dn}(x, m)$ .

The purpose of this paper is to propose general results about the existence of new solutions to real nonlinear equations, integrable or nonintegrable, continuous or discrete through the idea of parity-time reversal or PT symmetry. It may be noted here that in the last 15 years or so the idea of PT symmetry [5] has given us new insights. In quantum mechanics it has been shown that even if a Hamiltonian is not hermitian but if it is PT-invariant, then the energy eigenvalues are still real in case the PT symmetry is not broken spontaneously. Further, there has been a tremendous growth in the number of studies of open systems which are specially balanced by PT symmetry [6, 7, 8] in several PT-invariant open systems bearing both loss and gain. In particular, many researchers have obtained soliton solutions which have been shown to be stable within a certain parameter range [9, 10, 11].

It is worth specifying what exactly we mean by  $P$  and  $T$  and hence the PT symmetry. By  $P$  one means parity symmetry, i.e.  $x \rightarrow -x$ ,  $t \rightarrow t$  while by  $T$  one means time-reversal symmetry, i.e.  $t \rightarrow -t$ ,  $i \rightarrow -i$ ,  $x \rightarrow x$ . Thus by the combined PT symmetry we mean  $x \rightarrow -x$ ,  $t \rightarrow -t$ ,  $i \rightarrow -i$ .

In this context, recently we have highlighted one more novel aspect of PT symmetry. Specifically, we obtained new PT-invariant solutions of several real nonlinear equations with PT-eigenvalue  $+1$ . In particular, we showed [12] that if a real nonlinear equation admits soliton solutions in terms of  $\text{sech}x$  then

it also admits PT-invariant solutions  $\operatorname{sech} x \pm i \tanh x$  with PT-eigenvalue  $+1$ . We also showed that if a real nonlinear equation admits solutions in terms of  $\operatorname{sech}^2 x$  then it also admits PT-invariant solutions  $\operatorname{sech}^2 x \pm i \operatorname{sech} x \tanh x$  with PT-eigenvalue  $+1$ . In addition, we considered the periodic generalization of these results. It is worth pointing out that in all these cases the PT-invariant combinations (such as  $\operatorname{sech} x \pm i \tanh x$ ) are eigenfunctions of the PT operator with eigenvalue  $+1$ .

It is then natural to inquire if there are PT-invariant solutions with PT-eigenvalue  $-1$  and further in the case of coupled field theories, are there PT-invariant solutions with PT-eigenvalue  $-1$  in both the fields and also if there are mixed PT-invariant solutions with PT-eigenvalue  $+1$  in one field and  $-1$  in the other field. One of the aims of this paper is to provide answers to these questions. Our strategy will be to start with known real solutions and then make Ansätze for complex PT-invariant solutions and obtain conditions under which the Ansätze are valid. We show, through several examples (such as  $\phi^4$ ,  $\phi^6$ , sine-Gordon, double sine-Gordon, double sine-hyperbolic-Gordon equations), that whenever a real nonlinear equation, either continuous or discrete, integrable or nonintegrable, admits a solution in terms of  $\tanh \beta x$ , then it will necessarily also admit solutions in terms of the PT-invariant combinations  $\tanh 2\beta x \pm i \operatorname{sech} 2\beta x$  with PT-eigenvalue  $-1$  (i.e. with the PT-kink width being half of that of the corresponding real kink). Remarkably, in all these cases, the kink solution as well as the PT-invariant kink are solutions of the first order self-dual equation. Further in all these cases, the kink as well as the PT-invariant kink have the same topological charge and the same kink energy and both the solutions can be shown to be linearly stable. In view of this, we believe that the PT-invariant kink solutions may also find physical realization in coupled optical waveguides among other applications [13]. It is worth pointing out that some of the equations considered here have also been considered in their PT-symmetric deformed version [14].

We also generalize these results to the periodic case and show that whenever a nonlinear equation admits a solution in terms of  $\operatorname{sn}(x, m)$ , then it will necessarily also admit solutions in terms of the PT-invariant combinations  $\operatorname{sn}(x, m) \pm i \operatorname{cn}(x, m)$  as well as  $\operatorname{sn}(x, m) \pm i \operatorname{dn}(x, m)$  with PT-eigenvalue  $-1$ . Further, we also consider coupled field theory models and obtain PT-invariant solutions with PT-eigenvalue  $-1$  in both the fields (in addition to PT-eigenvalue  $+1$  in both the fields) and mixed solutions with PT-eigenvalue  $+1$  in one field and  $-1$  in the other field.

The plan of the paper is the following. In Sec. II we consider several self-dual first order equations which are known to admit topological kink solutions of the form  $\tanh \beta x$  and in all these cases show the existence of PT-invariant complex kink solutions of the form  $\tanh 2\beta x \pm i \operatorname{sech} 2\beta x$  with PT-eigenvalue  $-1$  and kink width being half of that of the corresponding real kink. We show that a given kink solution and the corresponding PT-invariant kink solution have the same topological charge, the same kink energy and further, both are linearly stable. For all the PT-invariant kink solutions we give explicit expressions for the zero mode. In Sec. III we consider four continuum and two discrete field theory models and show the existence of PT-invariant periodic kink solutions of the form  $\operatorname{sn}(x, m) \pm i \operatorname{cn}(x, m)$  and  $\operatorname{sn}(x, m) \pm i \operatorname{dn}(x, m)$  with PT-eigenvalue  $-1$  and also the corresponding hyperbolic PT-invariant kink solutions. In Sec. IV we discuss three coupled field theory models and show that these models not only admit PT-invariant solutions with PT-eigenvalue  $+1$  or  $-1$  for both the fields, but also mixed PT-invariant solutions with PT-eigenvalue  $+1$  in one field and  $-1$  in the other field. Section V is reserved for summary of the main results obtained where we also discuss some of the open problems.

## 2 PT-invariant Kink Solutions

We now discuss several examples from continuum field theories where a kink solution like  $\tanh \beta x$  is a solution of the first order self-dual equation. We consider several self-dual first order equations with known kink solutions and in all the cases obtain new PT-invariant solutions in terms of  $\tanh 2\beta x \pm i \operatorname{sech} 2\beta x$  with PT-eigenvalue  $-1$ .

As is well known, typically, the self-dual equations with kink solutions are of the form

$$\frac{d\phi}{dx} = \pm \sqrt{2V(\phi)}, \quad (1)$$

where  $V(\phi)$  has multiple degenerate minima and is positive semidefinite with its minimum value being zero at the degenerate minima. We show in general that both the kink topological charge as well as the kink energy remain unaltered, i.e. the usual kink solution and the new PT-invariant kink solutions have the same topological charge and the same kink energy. Further, we show that both the usual kink as well as the PT-invariant kink solutions are linearly stable and give explicit expressions for the nodeless zero mode

in all the cases.

## 2.1 $\phi^4$ Kink

The  $\phi^4$  field theory arises in several areas of physics. It has the celebrated kink solution which is the solution of the first order self-dual equation

$$\frac{d\phi}{dx} = \pm \sqrt{\frac{b}{2}} \left( \frac{a}{b} - \phi^2 \right). \quad (2)$$

We consider here and in rest of the examples, one of the self-dual equations. Exactly the same arguments are also valid for the other cases. It is well known that the kink solution to Eq. (2) is [15]

$$\phi_k = A \tanh \beta x, \quad (3)$$

provided

$$A = \sqrt{\frac{a}{b}}, \quad \beta = \sqrt{\frac{a}{2}}. \quad (4)$$

The corresponding topological charge is

$$Q = \phi(x = \infty) - \phi(x = -\infty) = 2\sqrt{\frac{a}{b}}, \quad (5)$$

while the corresponding kink energy is

$$\begin{aligned} E_k &= \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + V(\phi) \right] \\ &= \int_{-\sqrt{\frac{a}{b}}}^{\sqrt{\frac{a}{b}}} d\phi \sqrt{2V(\phi)} = \frac{4}{3} \sqrt{\frac{a^3}{b^3}}. \end{aligned} \quad (6)$$

Remarkably, even

$$\phi_{ptk} = A \tanh \beta_1 x + iB \operatorname{sech} \beta_1 x, \quad (7)$$

is an exact PT-invariant kink solution (with PT eigenvalue  $-1$ ) of the self-dual Eq. (2) provided

$$B = \pm A, \quad A = \sqrt{\frac{a}{b}}, \quad \beta_1 = \sqrt{2a} = 2\beta, \quad (8)$$

where  $\beta$  is as given by Eq. (4). Note that the corresponding topological charge as defined by Eq. (5) is the same for the kink solution (3) and the complex PT-invariant kink solution (7). This is because

$\phi(\pm\infty)$  remains unchanged. Further, even the kink energy is the same for the kink solution (3) and the PT-invariant kink solution (7) since as is clear from Eq. (6), the answer for the kink energy depends on  $V(\phi)$  and  $\phi(\pm\infty)$  both of which are the same for the kink solution (3) as well for the PT-invariant kink solution (7).

The arguments given above are rather general and hence it is clear that the topological charge and the kink energy are the same for any (real) kink solution and the corresponding complex PT-invariant kink solution. We have also explicitly checked it for all the cases mentioned below. Hence now onwards, for the remaining examples, we shall not discuss the kink topological charge and the kink energy for the PT-invariant kink solutions.

Let us now discuss the question of linear stability of the complex PT-invariant kink solution (7). In this context it is worth recalling that in the case of the standard kink solution as given by Eq. (3), the linear stability issue has been discussed a while ago [15] and it has been shown that on assuming

$$\phi = \phi_k + \eta(x)e^{i\omega t}, \quad (9)$$

and substituting it in the corresponding (second-order) field equation, one gets the stability equation

$$-\frac{d^2\eta(x)}{dx^2} + \frac{d^2V(\phi_k)}{dx^2}\eta(x) = \omega^2\eta(x). \quad (10)$$

Further, it is well known that the corresponding zero mode, i.e. the unnormalized ground state wave function  $\eta_0$  with  $\omega^2 = 0$  is given by

$$\eta_0(x) = \frac{d\phi_k}{dx}. \quad (11)$$

This discussion is easily generalized to the PT-invariant kink and in that case one gets the stability equation which is similar to (10) except that  $\frac{d^2V(\phi_k)}{dx^2}$  is replaced by  $\frac{d^2V(\phi_{ptk})}{dx^2}$  and the corresponding zero-mode is given by

$$\eta_{0,pt}(x) = \frac{d\phi_{ptk}}{dx}. \quad (12)$$

For the usual  $\phi^4$  kink, using Eq. (3), one gets the stability equation

$$-\frac{d^2\eta(x)}{dx^2} + [2a - 3a\text{sech}^2(\beta x)]\eta(x) = \omega^2\eta(x), \quad \beta = \sqrt{\frac{a}{2}}. \quad (13)$$

As is well known, this Schrödinger-like equation has two bound states with the corresponding eigenvalues and eigenfunctions being

$$\begin{aligned}\eta_0(x) &= \text{sech}^2(\beta x), \quad \omega_0^2 = 0, \\ \eta_1(x) &= \text{sech}(\beta x) \tanh(\beta x), \quad \omega_1^2 = \frac{3a}{2}.\end{aligned}\tag{14}$$

Note that while  $\eta_0$  is nodeless,  $\eta_1$  has one node.

Let us now discuss the stability of the PT-invariant kink solution (7). On using Eq. (7) in the stability Eq. (10), one gets a Schrödinger-like equation for the PT-invariant potential

$$-\frac{d^2\eta(x)}{dx^2} + [2a - 6a\text{sech}^2(\beta_1 x) \pm 6ia\text{sech}(\beta_1 x) \tanh(\beta_1 x)]\eta(x) = \omega^2\eta(x), \quad \beta_1 = \sqrt{2a}.\tag{15}$$

Remarkably, this Schrodinger-like PT-invariant equation too has two discrete states but unlike the real kink case, both the discrete states are nodeless but with different energies. In particular, the corresponding eigenvalues and eigenfunctions are

$$\begin{aligned}\eta_{0,pt}(x) &= \text{sech}(\beta_1 x) [\text{sech}(\beta_1 x) \mp i \tanh(\beta_1 x)] = \text{sech}(\beta_1 x) e^{\mp i \tan^{-1}[\sinh(\beta_1 x)]}, \quad \omega_0 = 0, \\ \eta_{1,pt}(x) &= \text{sech}^{1/2}(\beta_1 x) e^{\mp (3i/2) \tan^{-1}[\sinh(\beta_1 x)]}, \quad \omega_1^2 = \frac{3a}{2}.\end{aligned}\tag{16}$$

It is worth noting that the eigenenergies of the two bound states are identical (i.e.  $0, \sqrt{3a/2}$ ) for the usual and the PT-invariant kink solution.

Because of the linear stability of the PT-invariant kink solution, we suspect that the PT-invariant kink solution too can have physical relevance. It would be interesting to explore physical situations where such kink can indeed be relevant.

We now show that similar discussion is valid in the case of the other PT-invariant kink solutions and in all the cases the zero mode is nodeless thereby ensuring the linear stability of the PT-invariant kink solution.

## 2.2 $\phi^6$ Kinks

Unlike the  $\phi^4$  case, in the  $\phi^6$  case there are two different types of kink solutions depending on if one is at the first order transition point or below the transition point. We now show that in both the cases we have PT-invariant kink solutions.

**Case I:**  $T = T_{c1}$

In this case the kink solution satisfies the self dual equation

$$\frac{d\phi}{dx} = \pm\sqrt{\lambda}\phi(a^2 - \phi^2). \quad (17)$$

At this point there are two distinct kink solutions of the form  $\sqrt{1 \pm \tanh \beta x}$  and in both the cases the PT-invariant kink solutions also exist. For simplicity we only discuss one case, exactly the same arguments are also valid in the other case.

One of the well known kink solutions to Eq. (17) is [16]

$$\phi_k = A\sqrt{1 + \tanh \beta x}, \quad (18)$$

provided

$$A = a/\sqrt{2}, \quad \beta = \sqrt{\lambda}a^2. \quad (19)$$

The self-dual Eq. (17) also admits the PT-invariant kink solution

$$\phi_{ptk} = A\sqrt{1 + \tanh \beta_1 x \pm i \operatorname{sech} \beta_1 x}, \quad (20)$$

provided  $A$  is again as given by Eq. (19) while the inverse width  $\beta$  is again doubled, i.e.

$$\beta_1 = 2\sqrt{\lambda}a^2 = 2\beta, \quad (21)$$

where  $\beta$  is given by Eq. (19).

Let us now show that this PT-invariant kink solution is linearly stable. From Eq. (17) it is clear that

$$V(\phi) = \frac{\lambda}{2}\phi^2(a^2 - \phi^2)^2. \quad (22)$$

It is then easy to calculate  $\frac{d^2 V(\phi_{ptk})}{d\phi^2}$  and set up the stability equation for the PT-invariant kink solution (20). In particular, on setting  $y = \beta_1 x$ , the stability Eq. (10) takes the form

$$-\eta''(y) + (1/8)[5 - 15\operatorname{sech}^2 y + 3 \tanh y \pm 3i \operatorname{sech} y \pm 15i \operatorname{sech} y \tanh y]\eta(y) = \frac{\omega^2}{4\lambda a^4}\eta(y). \quad (23)$$

On using Eq. (12) the corresponding zero-mode turns out to be

$$\eta_{0,pt} \propto (\operatorname{sech} y)^{1/2} (1 - \tanh y)^{1/4} e^{\mp(3i/4) \tan^{-1}[\sinh y]}, \quad \omega_0 = 0. \quad (24)$$



**Case II:**  $T < T_{c1}$

In this case the self-dual equation is of the form

$$\frac{d\phi}{dx} = \pm \sqrt{\lambda}(a^2 - \phi^2) \sqrt{\phi^2 + b^2}. \quad (25)$$

The well known kink solution to this equation is [17, 18]

$$\phi = \frac{A \tanh \beta x}{\sqrt{1 - B^2 \tanh^2 \beta x}}, \quad (26)$$

provided

$$A = \frac{ab}{\sqrt{a^2 + b^2}}, \quad B = \frac{a^2}{a^2 + b^2}, \quad \beta = a\sqrt{\lambda(a^2 + b^2)}. \quad (27)$$

The solution (26) can be put in the form

$$\frac{\phi}{\sqrt{A^2 + B\phi^2}} = \tanh \beta x, \quad (28)$$

The self-dual Eq. (25) also admits the PT-invariant kink solution

$$\frac{\phi}{\sqrt{A^2 + B\phi^2}} = \tanh \beta_1 x \pm i \operatorname{sech} \beta_1 x, \quad (29)$$

provided  $A$  and  $B$  are as given by Eq. (27) while  $\beta$  is again doubled and given by

$$\beta_1 = 2a\sqrt{\lambda(a^2 + b^2)} = 2\beta, \quad (30)$$

where  $\beta$  is given by Eq. (27).

Let us now show that this PT-invariant kink solution is linearly stable. From Eq. (25) it is clear that

$$V(\phi) = \frac{\lambda}{2}(b^2 + \phi^2)(a^2 - \phi^2)^2. \quad (31)$$

It is then easy to calculate  $\frac{d^2 V(\phi_{ptk})}{d\phi^2}$  and set up the stability equation for the PT-invariant kink solution (29). In particular, on putting  $y = \beta_1 x$ , the stability Eq. (10) takes the form

$$\begin{aligned} -\eta''(y) + \frac{1}{4(a^2 + b^2)} \left( a^2 - 2b^2 + \frac{3b^2(b^2 - 2a^2)}{2a^2} \left[ \frac{(b^2/(b^2 + a^2) - 2\operatorname{sech}^2 y \pm 2i\operatorname{sech} y \tanh y)}{\operatorname{sech}^2 y + p^2} \right] \right. \\ \left. + \frac{15b^4}{16a^2} \left[ \frac{(b^2/(b^2 + a^2) - 2\operatorname{sech}^2 y \pm 2i\operatorname{sech} y \tanh y)}{\operatorname{sech}^2 y + p^2} \right]^2 \right) \eta(y) = \frac{\omega^2}{4a^2\lambda(a^2 + b^2)} \eta(y), \end{aligned} \quad (32)$$

where  $p^2 = \frac{b^4}{4a^2(a^2+b^2)}$ . On using Eq. (12), we obtain the well known zero-mode for the PT-invariant kink solution (29)

$$\eta_{0,pt} \propto \text{sech} y \sqrt{\frac{b^2}{a^2+b^2} - 2\text{sech}^2 y \mp 2i\text{sech} y \tanh y} \times \frac{[b^2/(a^2+b^2)](1 + \frac{b^2}{2a^2})\text{sech} y \tanh y \pm i(1+2p^2)\text{sech}^2 y \mp ip^2}{[\text{sech}^2 y + p^2]^2}. \quad (33)$$

### 2.3 Sine-Gordon Kink

The self-dual sine-Gordon equation is [15]

$$\frac{d\phi}{dx} = \pm 2 \sin \frac{\phi}{2}. \quad (34)$$

One of the well known kink solutions is

$$\phi = 4 \tan^{-1}(e^{-x}), \quad (35)$$

which can be put in the form

$$\cos(\phi/2) = \tanh \beta x, \quad \beta = 1. \quad (36)$$

The self-dual Eq. (34) also admits the PT-invariant kink solution

$$\cos(\phi/2) = \tanh \beta_1 x \pm i \text{sech} \beta_1 x, \quad (37)$$

provided  $\beta$  is again doubled, i.e.  $\beta_1 = 2 = 2\beta$  where  $\beta$  is given by Eq. (36).

Let us now show that this PT-invariant kink solution is linearly stable. From Eq. (34) it is clear that

$$V(\phi) = 1 - \cos(\phi). \quad (38)$$

It is then easy to calculate  $\frac{d^2 V(\phi_{ptk})}{d\phi^2}$  and set up the stability equation for the PT-invariant kink solution (37). In particular, on substituting  $y = \beta_1 x$  in Eq. (10) we obtain the stability equation

$$-\eta''(y) + \left[ \frac{1}{4} - \text{sech}^2 y \pm i \text{sech} y \tanh y \right] \eta(y) = \frac{\omega^2}{4} \eta(y). \quad (39)$$

Using Eq. (12), the zero-mode for the PT-invariant kink solution (37) turns out to be

$$\eta_{0,pt} \propto \sqrt{\text{sech} y} e^{\mp(i/2) \tan^{-1}(\sinh y)}. \quad (40)$$

## 2.4 Double sine-hyperbolic-Gordon Kink

The self-dual equation for the double sine-hyperbolic-Gordon (DSHG) equation is

$$\frac{d\phi}{dx} = \pm\sqrt{2}(\zeta \cosh 2\phi - n). \quad (41)$$

The well known kink solution in this case is [19, 20, 21]

$$\phi = \tanh^{-1} \left[ \sqrt{\frac{n-\zeta}{n+\zeta}} \tanh \beta x \right], \quad \beta = \sqrt{n^2 - \zeta^2}, \quad (42)$$

which can be put in the form

$$\tanh \phi = \sqrt{\frac{n-\zeta}{n+\zeta}} \tanh \beta x. \quad (43)$$

The same self-dual Eq. (41) also admits the PT-invariant kink solution

$$\tan \phi = \sqrt{\frac{n-\zeta}{n+\zeta}} [\tanh \beta_1 x \pm i \operatorname{sech} \beta_1 x], \quad (44)$$

provided  $\beta$  is doubled, i.e.  $\beta_1 = 2\sqrt{n^2 - \zeta^2} = 2\beta$  where  $\beta$  is given by Eq. (42).

Let us now show that this PT-invariant kink solution is linearly stable. From Eq. (41) it is clear that

$$V(\phi) = (\zeta \cosh 2\phi - n)^2. \quad (45)$$

It is then easy to calculate  $\frac{d^2 V(\phi_{ptk})}{d\phi^2}$  and set up the stability equation for the PT-invariant kink solution (44). In particular, on substituting  $y = \beta_1 x$  in Eq. (10) we obtain the stability equation

$$\begin{aligned} -\eta''(y) + \frac{1}{n-\zeta} \left( 2\zeta - \frac{2\zeta}{(n^2 \operatorname{sech}^2 y + \zeta^2 \tanh^2 y)} [2\zeta + (n+4\zeta)(n \operatorname{sech}^2 y + \zeta \tanh^2 y \pm i(n-\zeta) \operatorname{sech} 6 \tanh y] \right. \\ \left. + \frac{8\zeta^2(n+\zeta)(n \operatorname{sech}^2 y + \zeta \tanh^2 y)}{(n^2 \operatorname{sech}^2 y + \zeta^2 \tanh^2 y)^2} [n \operatorname{sech}^2 y + \zeta \tanh^2 y \pm i(n-\zeta) \operatorname{sech} y \tanh y] \right) \eta(y) = \frac{\omega^2}{4(n^2 - \zeta^2)} \eta(y). \end{aligned} \quad (46)$$

Using Eq. (12), the zero-mode for the PT-invariant kink solution (44) turns out to be

$$\eta_{0,pt} \propto \frac{\operatorname{sech} y (n \operatorname{sech} y \mp i\zeta \tanh y)}{n^2 \operatorname{sech}^2 y + \zeta^2 \tanh^2 y}. \quad (47)$$

## 2.5 Double Sine-Gordon Kink

Consider the following self-dual equation for the double sine-Gordon case

$$\frac{d\phi}{dx} = \pm\sqrt{2\lambda} \left( \sin \phi - \frac{\mu}{\lambda} \right), \quad \mu < \lambda. \quad (48)$$

In this case the well known kink solution is [22]

$$\phi = 2 \tan^{-1} \left( \sqrt{\frac{\lambda - \mu}{\lambda + \mu}} \tanh \beta x \right) + \frac{\pi}{2}, \quad (49)$$

which can be put in the form

$$\tan \left( \frac{\phi}{2} - \frac{\pi}{4} \right) = \sqrt{\frac{\lambda - \mu}{\lambda + \mu}} \tanh(\beta x), \quad (50)$$

provided

$$\beta = \sqrt{\frac{\lambda(1 - \frac{\mu^2}{\lambda^2})}{2}}. \quad (51)$$

The same self-dual Eq. (48) admits the PT-invariant kink solution

$$\tan \left( \frac{\phi}{2} - \frac{\pi}{4} \right) = \sqrt{\frac{\lambda - \mu}{\lambda + \mu}} [\tanh \beta x \pm i \operatorname{sech} \beta x], \quad (52)$$

provided  $\beta$  is doubled, i.e.

$$\beta_1 = \sqrt{2\lambda \left( 1 - \frac{\mu^2}{\lambda^2} \right)} = 2\beta, \quad (53)$$

where  $\beta$  is given by Eq. (51).

Let us now show that this PT-invariant kink solution is linearly stable. From Eq. (48) it is clear that

$$V(\phi) = \lambda \left( \sin \phi - \frac{\mu}{\lambda} \right)^2. \quad (54)$$

It is then easy to calculate  $\frac{d^2 V(\phi_{ptk})}{d\phi^2}$  and set up the stability equation for the PT-invariant kink solution (52). In particular, on substituting  $y = \beta_1 x$  in Eq. (10) we obtain the stability equation

$$\begin{aligned} -\eta''(y) + \frac{\lambda^2}{\lambda^2 - \mu^2} \left[ 1 + \frac{\mu(\mu\lambda \mp i(\lambda^2 - \mu^2)\operatorname{sech} y)}{\lambda(\mu^2 \operatorname{sech}^2 y + \lambda^2 \tanh^2 y)} \right. \\ \left. - 2 \frac{\mu^2 \lambda^2 - (\lambda^2 - \mu^2)^2 \operatorname{sech}^2 y \mp 2i\mu\lambda(\lambda^2 - \mu^2)\operatorname{sech} y}{(\mu^2 \operatorname{sech}^2 y + \lambda^2 \tanh^2 y)^2} \right] \eta(y) = \frac{\omega^2 \lambda}{2(\lambda^2 - \mu^2)} \eta(y). \end{aligned} \quad (55)$$

Using Eq. (12), the zero-mode for the PT-invariant kink solution (52) turns out to be

$$\eta_{0,pt} \propto \frac{\operatorname{sech} y (\mu \operatorname{sech} y \mp i \lambda \tanh y)}{\mu^2 \operatorname{sech}^2 y + \lambda^2 \tanh^2 y}. \quad (56)$$

### 3 PT-Invariant Periodic Kink Solutions

We now discuss a few examples from both the continuum and the discrete field theories where both periodic and hyperbolic kink-like solutions are known, and show that in all these cases one also has complex PT-invariant periodic as well as hyperbolic kink solutions.

#### 3.1 mKdV Equation

We first discuss the celebrated mKdV equation

$$u_t + u_{xxx} - 6u^2u_x = 0, \quad (57)$$

which is a well known integrable equation having application in several areas [23]. One of the exact, periodic solutions to the mKdV Eq. (57) is

$$u = A\sqrt{m}\operatorname{sn}[\beta(x - vt), m], \quad (58)$$

provided

$$A^2 = \beta^2, \quad v = -(1 + m)\beta^2. \quad (59)$$

In the limit  $m = 1$ , the solution (58) goes over to the hyperbolic kink solution

$$u = A \tanh[\beta(x - vt)], \quad (60)$$

and in this case  $v = -2\beta^2$ .

Remarkably, even the complex PT-invariant combination (with PT eigenvalue  $-1$ )

$$u = A\sqrt{m}\operatorname{sn}[\beta(x - vt), m] + iB\sqrt{m}\operatorname{cn}[\beta(x - vt), m], \quad (61)$$

is an exact solution to the mKdV Eq. (57) provided

$$B = \pm A, \quad \beta^2 = 4A^2, \quad v = -\frac{(2 - m)}{2}\beta^2. \quad (62)$$

Yet another PT-invariant solution (with PT eigenvalue  $-1$ ) is

$$u = A\sqrt{m}\operatorname{sn}[\beta(x - vt), m] + iB\operatorname{dn}[\beta(x - vt), m], \quad (63)$$

provided

$$B = \pm A, \quad \beta^2 = 4A^2, \quad v = -\frac{(2m-1)}{2}\beta^2. \quad (64)$$

We thus have two new periodic solutions of the mKdV Eq. (57). In the limit  $m = 1$ , both these solutions go over to the hyperbolic PT-invariant solution

$$u = A \tanh[\beta(x - vt)] \pm iB \operatorname{sech}[\beta(x - vt)], \quad (65)$$

provided

$$B = \pm A, \quad \beta^2 = 4A^2, \quad v = -(1/2)\beta^2. \quad (66)$$

There is also a complex PT-invariant solution to the mKdV Eq. (57) with PT-eigenvalue  $+1$ . Let us first note that the mKdV Eq. (57) has an exact solution

$$u = \frac{A\sqrt{m}\operatorname{cn}[\beta(x - vt), m]}{\operatorname{dn}[\beta(x - vt), m]}, \quad (67)$$

provided

$$A^2 = \beta^2, \quad v = -(1 + m)\beta^2. \quad (68)$$

It is easily checked that the same Eq. (57) also has the complex PT invariant solution with PT-eigenvalue  $+1$

$$u = A\sqrt{m} \frac{\operatorname{cn}[\beta(x - vt), m]}{\operatorname{dn}[\beta(x - vt), m]} + iB\sqrt{m(1 - m)} \frac{\operatorname{sn}[\beta(x - vt), m]}{\operatorname{dn}[\beta(x - vt), m]}, \quad (69)$$

provided

$$B = \pm A, \quad \beta^2 = 4A^2, \quad v = -\frac{(2 - m)}{2}\beta^2. \quad (70)$$

Before completing this subsection, we would like to note that in our recent paper [12] we had considered the other mKdV equation, i.e.

$$u_t + u_{xxx} + 6u^2u_x = 0, \quad (71)$$

and had shown that in that case one has complex PT-invariant solutions of the form  $\operatorname{cn}(x, m) \pm i\operatorname{sn}(x, m)$  and  $\operatorname{dn}(x, m) \pm i\operatorname{sn}(x, m)$  with PT-eigenvalue  $+1$ . We now show that Eq. (71) also has a PT-invariant solution with PT-eigenvalue  $-1$ . Let us first note that

$$u = A\sqrt{m(1 - m)} \frac{\operatorname{sn}[\beta(x - vt), m]}{\operatorname{dn}[\beta(x - vt), m]}, \quad (72)$$

is an exact solution to Eq. (71) provided

$$A^2 = \beta^2, \quad v = (2m - 1)\beta^2. \quad (73)$$

It is easily checked that the same Eq. (71) also has the complex PT invariant solution with PT-eigenvalue  $-1$

$$u = A\sqrt{m(1-m)}\frac{\text{sn}[\beta(x-vt), m]}{\text{dn}[\beta(x-vt), m]} + iB\sqrt{m}\frac{\text{cn}[\beta(x-vt), m]}{\text{dn}[\beta(x-vt), m]}, \quad (74)$$

provided

$$B = \pm A, \quad 4A^2 = \beta^2, \quad v = -\frac{(2-m)}{2}\beta^2. \quad (75)$$

### 3.2 $\phi^4$ Field Theory

The field equation in this case is

$$\phi_{xx} = a\phi + b\phi^3, \quad (76)$$

which also follows from the self-dual first order Eq. (2). We now show that in this case too one has complex PT-invariant solutions with PT eigenvalue  $-1$ .

One of the exact, periodic solutions to the  $\phi^4$  Eq. (76) is [24]

$$u = A\sqrt{m}\text{sn}(\beta x, m), \quad (77)$$

provided

$$bA^2 = 2\beta^2, \quad a = -(1+m)\beta^2. \quad (78)$$

In the limit  $m = 1$ , the solution (77) goes over to the hyperbolic kink solution discussed in the previous section.

Remarkably, even the complex PT-invariant combination (with PT eigenvalue  $-1$ )

$$u = A\sqrt{m}\text{sn}(\beta x, m) + iB\sqrt{m}\text{cn}(\beta x, m), \quad (79)$$

is an exact solution to Eq. (76) provided

$$B = \pm A, \quad \beta^2 = 2bA^2, \quad a = -\frac{(2m-1)}{2}\beta^2. \quad (80)$$

Yet another PT-invariant solution (with PT eigenvalue  $-1$ ) is

$$u = A\sqrt{m}\operatorname{sn}(\beta x, m) + iB\operatorname{dn}(\beta x, m), \quad (81)$$

provided

$$B = \pm A, \quad \beta^2 = 2bA^2, \quad a = -\frac{(2-m)}{2}\beta^2. \quad (82)$$

In the limit  $m = 1$ , both solutions (79) and (81) go over to the hyperbolic PT-invariant kink solution discussed in the previous section.

Another periodic solution to Eq. (76) is

$$\phi = A\sqrt{m(1-m)}\frac{\operatorname{sn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \quad (83)$$

provided

$$bA^2 = -2\beta^2, \quad a = -(2m-1)\beta^2. \quad (84)$$

The complex PT-invariant combination (with PT-eigenvalue  $-1$ )

$$\phi = A\sqrt{m(1-m)}\frac{\operatorname{sn}(\beta x, m)}{\operatorname{dn}(\beta x, m)} + iB\sqrt{m}\frac{\operatorname{cn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \quad (85)$$

is also an exact solution to Eq. (76) provided

$$B = \pm A, \quad 2bA^2 = -\beta^2, \quad a = -\frac{(2-m)}{2}\beta^2. \quad (86)$$

So far we have discussed complex PT-invariant solutions (with PT-eigenvalue  $-1$ ) of the  $\phi^4$  field Eq. (76). Further, in a recent paper we have already obtained complex PT-invariant periodic solutions of the  $\phi^4$  field Eq. (76) with PT-eigenvalue  $+1$ . They were of the form  $\operatorname{cn}(x, m) \pm i\operatorname{sn}(x, m)$  and  $\operatorname{dn}(x, m) \pm i\operatorname{sn}(x, m)$ . We now show that the same Eq. (76) also has another periodic PT-invariant solution with PT-eigenvalue  $+1$ . Let us first note that one of the exact solutions to Eq. (76) is

$$\phi = A\sqrt{m}\frac{\operatorname{cn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \quad (87)$$

provided

$$bA^2 = 2\beta^2, \quad a = -(1+m)\beta^2. \quad (88)$$



The complex PT-invariant combination (with PT-eigenvalue +1)

$$\phi = A\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} + iB\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)}, \quad (89)$$

is also an exact solution to Eq. (76) provided

$$B = \pm A, \quad 2bA^2 = \beta^2, \quad a = -\frac{2-m}{2}\beta^2. \quad (90)$$

### 3.3 KdV Equation

In our recent publication [12] we have also obtained complex PT-invariant solutions of the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0, \quad (91)$$

with PT eigenvalue +1. We now discuss one more complex PT-invariant periodic solution of this equation with PT-eigenvalue +1. To begin with, it is easy to check that one of the exact periodic solutions of the KdV Eq. (91) is

$$u = \frac{A(1-m)}{\text{dn}^2[\beta(x-vt), m]}, \quad (92)$$

provided

$$A = -2\beta^2, \quad v = 4(2-m)\beta^2. \quad (93)$$

The same equation also admits the complex, PT-invariant, periodic solution

$$u = \frac{A(1-m)}{\text{dn}^2[\beta(x-vt), m]} + iB\sqrt{1-m}\frac{m\text{sn}[\beta(x-vt), m]\text{cn}[\beta(x-vt), m]}{\text{dn}^2[\beta(x-vt), m]}, \quad (94)$$

provided

$$B = \pm A, \quad A = -\beta^2, \quad v = (2-m)\beta^2. \quad (95)$$

It may be noted that (94) is a nonsingular, periodic solution which vanishes in the hyperbolic limit  $m = 1$ .

### 3.4 $\phi^3$ Field Theory

This field theory arises in the context of third order phase transitions [25] and is also relevant to tachyon condensation [26]. In our recent publication [12] we also discussed complex PT-invariant periodic solutions

of the  $\phi^3$  field theory with PT-eigenvalue +1. In this subsection, we discuss one more complex, PT-invariant, periodic solution with PT-eigenvalue +1. We first notice that one of the exact solutions of the  $\phi^3$  field theory

$$\phi_{xx} = a\phi + b\phi^2, \quad (96)$$

is

$$\phi = \frac{A(1-m)}{\text{dn}^2[\beta(x), m]} + Ay, \quad (97)$$

provided

$$bA = -6\beta^2, \quad a = 4[2 - m + 3y]\beta^2, \quad y = \frac{-(2-m) \pm \sqrt{1-m+m^2}}{3}. \quad (98)$$

The same Eq. (96) also admits the complex, PT-invariant periodic solution (with PT-eigenvalue +1)

$$\phi = \frac{A(1-m)}{\text{dn}^2[\beta(x), m]} + Ay + iBm\sqrt{1-m}\frac{\text{cn}[\beta(x), m]\text{sn}[\beta(x), m]}{\text{dn}^2[\beta(x), m]}, \quad (99)$$

provided

$$B = \pm A, \quad bA = -3\beta^2, \quad a = (2-m+6y)\beta^2, \quad y = \frac{-(2-m) \pm \sqrt{16-16m+m^2}}{6}. \quad (100)$$

Note that this is a nonsingular, complex, PT-invariant solution which vanishes in the hyperbolic limit  $m = 1$ .

### 3.5 Discrete $\phi^4$ Equation

We now discuss two discrete models and show that both these models also admit PT-invariant periodic and hyperbolic kink solutions. Let us consider the discrete  $\phi^4$  equation

$$\frac{1}{h^2}[\phi_{n+1} + \phi_{n-1} - 2\phi_n] + a\phi_n - \frac{\lambda}{2}\phi_n^2[\phi_{n+1} + \phi_{n-1}] = 0. \quad (101)$$

It is easy to check that Eq. (101) admits an exact periodic kink solution [27]

$$\phi_n = A\sqrt{m}\text{sn}(\beta n, m), \quad (102)$$

provided

$$A^2 = \frac{2\text{sn}^2(\beta, m)}{h^2\lambda}, \quad ah^2 = 2[1 - \text{cn}(\beta, m)\text{dn}(\beta, m)]. \quad (103)$$

The same model also admits a PT-invariant periodic kink solution

$$\phi_n = A\sqrt{m}\operatorname{sn}(\beta n, m) + iB\sqrt{m}\operatorname{cn}(\beta n, m), \quad (104)$$

provided

$$B = \pm A, \quad A^2 = \frac{2\operatorname{sn}^2(\beta, m)}{h^2\lambda[1 + \operatorname{dn}(\beta, m)]^2}, \quad ah^2 = 2 \left[ 1 - \frac{2\operatorname{cn}(\beta, m)}{1 + \operatorname{dn}(\beta, m)} \right]. \quad (105)$$

Further, the model also admits yet another PT-invariant periodic kink solution

$$\phi_n = A\sqrt{m}\operatorname{sn}(\beta n, m) + iB\operatorname{dn}(\beta n, m), \quad (106)$$

provided

$$B = \pm A, \quad A^2 = \frac{2\operatorname{sn}^2(\beta, m)}{h^2\lambda[1 + \operatorname{cn}(\beta, m)]^2}, \quad ah^2 = 2 \left[ 1 - \frac{2\operatorname{dn}(\beta, m)}{1 + \operatorname{cn}(\beta, m)} \right]. \quad (107)$$

In the limit  $m = 1$ , both the solutions (104) and (106) go over to the hyperbolic PT-invariant kink solution

$$\phi_n = A \tanh(\beta n) + iB \operatorname{sech}(\beta n), \quad (108)$$

provided

$$B = \pm A, \quad A^2 = \frac{2 \tanh^2(\frac{\beta}{2})}{h^2\lambda}, \quad ah^2 = 2 \tanh^2\left(\frac{\beta}{2}\right). \quad (109)$$

While deriving results in this and the next subsection, we have made use of several not so well known identities satisfied by the Jacobi elliptic functions [28].

### 3.6 Discrete mKdV Equation

Let us consider the discrete mKdV equation

$$\frac{du_n}{dt} + \alpha(u_{n+1} - u_{n-1}) - \lambda u_n^2(u_{n+1} - u_{n-1}) = 0. \quad (110)$$

It is easily checked that this model admits the periodic kink solution [29]

$$u_n = A\sqrt{m}\operatorname{sn}[\beta(n - vt), m], \quad (111)$$

provided

$$\lambda A^2 = \alpha \operatorname{sn}^2(\beta, m), \quad \beta v = 2\alpha \operatorname{sn}(\beta, m). \quad (112)$$

Remarkably, the same model (110) also admits a complex PT-invariant periodic kink solution

$$u_n = A\sqrt{m}\operatorname{sn}[\beta(n-vt), m] + iB\sqrt{m}\operatorname{cn}[\beta(n-vt), m], \quad (113)$$

provided

$$B = \pm A, \quad \lambda A^2 = \frac{\alpha \operatorname{sn}^2(\beta, m)}{[1 + \operatorname{dn}(\beta, m)]^2}, \quad \beta v = \frac{4\alpha \operatorname{sn}(\beta, m)}{1 + \operatorname{dn}(\beta, m)}. \quad (114)$$

Further, the same model also admits yet another complex PT-invariant periodic kink solution

$$u_n = A\sqrt{m}\operatorname{sn}[\beta(n-vt), m] + iB\operatorname{dn}[\beta(n-vt), m], \quad (115)$$

provided

$$B = \pm A, \quad \lambda A^2 = \frac{\alpha \operatorname{sn}^2(\beta, m)}{[1 + \operatorname{cn}(\beta, m)]^2}, \quad \beta v = \frac{4\alpha \operatorname{sn}(\beta, m)}{1 + \operatorname{cn}(\beta, m)}. \quad (116)$$

In the limit  $m = 1$ , both the complex PT-invariant solutions (113) and (115) go over to the complex PT-invariant hyperbolic kink solution

$$u_n = A \tanh(\beta n) + iB \operatorname{sech}(\beta n), \quad (117)$$

provided

$$B = \pm A, \quad \lambda A^2 = \alpha \tanh^2(\beta/2), \quad \beta v = 4\alpha \tanh(\beta/2). \quad (118)$$

## 4 PT-Invariant Solutions in Three Coupled models

We now consider three coupled models and show that in all these cases one has PT-invariant solutions for the coupled fields. In particular, we show that these models admit three different types of complex, PT-invariant periodic as well as hyperbolic solutions. In particular, there are solutions with PT eigenvalue  $+1$  or  $-1$  in both the fields and also solutions with PT eigenvalue  $+1$  in one field and  $-1$  in the other field.

### 4.1 Coupled $\phi^4$ Model

We first consider a coupled  $\phi^4$  model

$$\phi_{xx} = a_1\phi + b_1\phi^3 + \alpha\phi\psi^2, \quad (119)$$

$$\psi_{xx} = a_2\psi + b_2\psi^3 + \alpha\psi\phi^2, \quad (120)$$

and show that in this case all three types (i.e. those with PT-eigenvalue  $+1$  or  $-1$  in both the fields or  $+1$  in one field and  $-1$  in the other field) of PT-invariant periodic as well as hyperbolic solutions are allowed. We shall see that there are solutions not only in terms of Lamé polynomials of order one but also in terms of Lamé polynomials of order two.

#### 4.1.1 Solutions in Terms of Lamé Polynomials of order one

Let us first discuss solutions in terms of Lamé polynomials of order one.

##### Case I: Solutions With PT Eigenvalue $-1$ in Both The Fields

We first discuss PT-invariant solutions with PT eigenvalue  $-1$  in both the fields.

It is easy to check that one of the exact solutions to Eq. (119) is [30]

$$\phi = A\sqrt{m}\operatorname{sn}[\beta x, m], \quad \psi = B\sqrt{m}\operatorname{sn}[\beta x, m], \quad (121)$$

provided

$$b_1A^2 + \alpha B^2 = b_2B^2 + \alpha A^2 = 2\beta^2, \quad a_1 = a_2 = -(1+m)\beta^2. \quad (122)$$

The same coupled model also admits the PT-invariant periodic solution

$$\begin{aligned} \phi &= A\sqrt{m}\operatorname{sn}[\beta x, m] + iD\sqrt{m}\operatorname{cn}[\beta x, m], \\ \psi &= B\sqrt{m}\operatorname{sn}[\beta x, m] + iF\sqrt{m}\operatorname{cn}[\beta x, m], \end{aligned} \quad (123)$$

provided

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{(2-m)\beta^2}{2}, \quad (124)$$

and further

$$2(b_1A^2 + \alpha B^2) = 2(b_2B^2 + \alpha A^2) = \beta^2. \quad (125)$$

Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated.

Further, the same model also admits another PT-invariant periodic solution

$$\begin{aligned} \phi &= A\sqrt{m}\operatorname{sn}[\beta x, m] + iD\operatorname{dn}[\beta x, m], \\ \psi &= B\sqrt{m}\operatorname{sn}[\beta x, m] + iF\operatorname{dn}[\beta x, m], \end{aligned} \quad (126)$$

provided

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{(2m-1)\beta^2}{2}, \quad (127)$$

and if Eq. (125) is satisfied. Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated.

In the limit  $m = 1$ , both the periodic PT-invariant solutions (123) and (126) go over to the coupled hyperbolic PT-invariant solution

$$\begin{aligned} \phi &= A \tanh(\beta x) + iD \operatorname{sech}(\beta x), \\ \psi &= B \tanh(\beta x) + iF \operatorname{sech}(\beta x), \end{aligned} \quad (128)$$

provided Eq. (125) is satisfied and if further

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{\beta^2}{2}. \quad (129)$$

On solving Eq. (125) we find that

$$A^2 = \frac{(b_2 - \alpha)\beta^2}{2(b_1 b_2 - \alpha^2)}, \quad B^2 = \frac{(b_1 - \alpha)\beta^2}{2(b_1 b_2 - \alpha^2)}, \quad (130)$$

provided  $b_1 b_2 \neq \alpha^2$ . In the special case when  $b_1 b_2 = \alpha^2$  which implies  $b_1 = b_2 = \alpha$ , instead of the two relations of Eq. (125), we only have a single relation

$$2b_1(A^2 + B^2) = \beta^2, \quad (131)$$

and hence, in this case,  $A, B$  are indeterminate except that they satisfy the constraint (131).

Yet another exact solution to Eqs. (119), (120) is

$$\begin{aligned} \phi &= A\sqrt{m(1-m)} \frac{\operatorname{sn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \\ \psi &= B\sqrt{m(1-m)} \frac{\operatorname{sn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \end{aligned} \quad (132)$$

provided

$$b_1 A^2 + \alpha B^2 = b_2 B^2 + \alpha A^2 = -2\beta^2, \quad a_1 = a_2 = (2m-1)\beta^2. \quad (133)$$

Remarkably, we find that the same coupled model also admits the PT-invariant periodic solution

$$\begin{aligned} \phi &= A\sqrt{m(1-m)} \frac{\operatorname{sn}(\beta x, m)}{\operatorname{dn}(\beta x, m)} + iD\sqrt{m} \frac{\operatorname{cn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \\ \psi &= B\sqrt{m(1-m)} \frac{\operatorname{sn}(\beta x, m)}{\operatorname{dn}(\beta x, m)} + iF\sqrt{m} \frac{\operatorname{cn}(\beta x, m)}{\operatorname{dn}(\beta x, m)}, \end{aligned} \quad (134)$$

with PT-eigenvalue  $-1$  in both the fields provided

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{(2-m)\beta^2}{2}, \quad (135)$$

and further

$$2(b_1 A^2 + \alpha B^2) = 2(b_2 B^2 + \alpha A^2) = -\beta^2. \quad (136)$$

Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated. On solving Eq. (136) we find that

$$A^2 = -\frac{(b_2 - \alpha)\beta^2}{2(b_1 b_2 - \alpha^2)}, \quad B^2 = -\frac{(b_1 - \alpha)\beta^2}{2(b_1 b_2 - \alpha^2)}, \quad (137)$$

provided  $b_1 b_2 \neq \alpha^2$ . In the special case when  $b_1 b_2 = \alpha^2$  which implies  $b_1 = b_2 = \alpha$ , instead of the two relations of Eq. (136), we only have a single relation

$$2b_1(A^2 + B^2) = -\beta^2, \quad (138)$$

and hence, in this case,  $A, B$  are indeterminate except that they satisfy the constraint (138).

## Case II: Solutions with Mixed PT Eigenvalues

We now discuss mixed PT-invariant solutions, i.e. PT-invariant solutions with PT eigenvalue  $+1$  in one field and  $-1$  in the other field.

It is easy to check that one of the exact solutions to Eqs. (119), (120) is

$$\phi = A\sqrt{m}\text{sn}[\beta x, m], \quad \psi = B\sqrt{m}\text{cn}[\beta x, m], \quad (139)$$

provided

$$b_1 A^2 - \alpha B^2 = \alpha A^2 - b_2 B^2 = 2\beta^2, \quad a_1 + m\alpha B^2 = -(1+m)\beta^2, \quad a_2 + m\alpha A^2 = (2m-1)\beta^2. \quad (140)$$

We find that the same coupled model also admits the mixed PT-invariant periodic solution

$$\begin{aligned} \phi &= A\sqrt{m}\text{sn}[\beta x, m] + iD\sqrt{m}\text{cn}[\beta x, m], \\ \psi &= B\sqrt{m}\text{cn}[\beta x, m] + iF\sqrt{m}\text{sn}[\beta x, m], \end{aligned} \quad (141)$$

provided

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{(2-m)\beta^2}{2}, \quad (142)$$

and further

$$2(b_1 A^2 - \alpha B^2) = 2(\alpha A^2 - b_2 B^2) = \beta^2. \quad (143)$$

Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated.

It is easy to check that one of the exact solutions to Eqs. (119), (120) is

$$\phi = A\sqrt{m}\operatorname{sn}[\beta x, m], \quad \psi = B\operatorname{dn}[\beta x, m], \quad (144)$$

provided

$$b_1 A^2 - \alpha B^2 = \alpha A^2 - b_2 B^2 = 2\beta^2, \quad a_1 + \alpha B^2 = -(1+m)\beta^2, \quad a_2 + \alpha A^2 = (2-m)\beta^2. \quad (145)$$

The same model also admits another PT-invariant periodic solution

$$\begin{aligned} \phi &= A\sqrt{m}\operatorname{sn}[\beta x, m] + iD\operatorname{dn}[\beta x, m], \\ \psi &= B\operatorname{dn}[\beta x, m] + iF\sqrt{m}\operatorname{sn}[\beta x, m], \end{aligned} \quad (146)$$

provided

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{(2m-1)\beta^2}{2}, \quad (147)$$

and further if Eq. (143) is satisfied. Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated.

In the limit  $m = 1$ , both the periodic PT-invariant solutions (142) and (146) go over to the coupled, hyperbolic, mixed PT-invariant solution

$$\begin{aligned} \phi &= A \tanh(\beta x) + iD \operatorname{sech}(\beta x), \\ \psi &= B \operatorname{sech}(\beta x) + iF \tanh(\beta x), \end{aligned} \quad (148)$$

provided Eq. (143) is satisfied and if further

$$D = \pm A, \quad F = \pm B, \quad a_1 = a_2 = -\frac{\beta^2}{2}. \quad (149)$$

On solving Eq. (143) we find that

$$A^2 = \frac{(b_2 - \alpha)\beta^2}{2(b_1 b_2 - \alpha^2)}, \quad B^2 = \frac{(\alpha - b_1)\beta^2}{2(b_1 b_2 - \alpha^2)}, \quad (150)$$



provided  $b_1 b_2 \neq \alpha^2$ . In the special case when  $b_1 b_2 = \alpha^2$  which implies  $b_1 = b_2 = \alpha$ , instead of the two relations of Eq. (143), we only have a single relation

$$2b_1(A^2 - B^2) = \beta^2. \quad (151)$$

Thus  $A, B$  are indeterminate in this case except that they must satisfy the constraint (151).

Yet another periodic solution to the coupled Eqs. (119), (120) is

$$\begin{aligned} \phi &= A\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)}, \\ \psi &= B\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)}, \end{aligned} \quad (152)$$

provided Eq. (140) is satisfied.

We find that the same coupled model also admits the PT-invariant periodic solution

$$\begin{aligned} \phi &= A\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} + iD\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)}, \\ \psi &= B\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)} + iF\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)}, \end{aligned} \quad (153)$$

provided Eq. (143) is satisfied and if further

$$D = \pm A, \quad F = \mp B, \quad a_1 = a_2 = -\frac{(2-m)\beta^2}{2}. \quad (154)$$

Note that the signs of  $D = \pm A$  and  $F = \mp B$  are correlated.

### Case III: Solutions With PT Eigenvalue +1 in Both The Fields

We now discuss complex PT-invariant periodic solutions with PT-eigenvalue +1 in both the fields. In our earlier paper we have already discussed PT-invariant solutions of the form  $\text{cn}(x, m) \pm i\text{sn}(x, m)$  and  $\text{dn}(x, m) \pm i\text{sn}(x, m)$  with PT-eigenvalue +1. We now show that there is another PT-invariant solution with PT-eigenvalue +1 in both the fields.

One exact solution to Eqs. (119), (120) is

$$\phi = A\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)}, \quad \psi = B\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)}, \quad (155)$$

provided Eq. (122) is satisfied. We find that the same coupled model also admits the PT-invariant periodic solution with PT-eigenvalue +1 in both the fields

$$\begin{aligned}\phi &= A\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} + iD\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)}, \\ \psi &= B\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}(\beta x, m)} + iF\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}(\beta x, m)},\end{aligned}\tag{156}$$

provided Eqs. (124) and (125) are satisfied.

#### 4.1.2 Solutions in Terms of Lamé Polynomials of Order Two

We now show that for the coupled  $\phi^4$  model (119), (120) one has all three types (i.e. those with PT-eigenvalue +1 or -1 for both the fields or +1 in one field and -1 in the other field) of PT-invariant solutions are allowed in terms of Lamé polynomials of order two.

##### Case I: Solutions With PT Eigenvalue -1 in Both The Fields

It is easy to check that

$$\phi = Am\frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)}, \quad \psi = B\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)},\tag{157}$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad b_1 A^2 = -6(1-m)\beta^2, \quad a_1 = (5m-4)\beta^2, \quad a_2 = (5m-1)\beta^2.\tag{158}$$

Remarkably, the PT-invariant combination with PT-eigenvalue -1

$$\begin{aligned}\phi &= Am\frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} + iD\left[\frac{1-m}{\text{dn}^2(\beta x, m)} + y\right] \\ \psi &= B\sqrt{m(1-m)}\frac{\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} + iF\sqrt{m}\frac{\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)},\end{aligned}\tag{159}$$

is also an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad D = \pm A, \quad F = \mp B, \quad 2b_1 A^2 y = 3\beta^2,\tag{160}$$

$$a_1 = b_1 A^2 + [2 - m + (9/2)y]\beta^2, \quad a_2 = b_1 A^2 + [2 - m + (3/2)y]\beta^2,\tag{161}$$

and  $y$  is given by Eq. (100).

## Case II: Solutions With Mixed PT Eigenvalues

Now let us discuss the so called mixed PT-invariant solutions, i.e. those with PT-eigenvalue  $+1$  in one field and  $-1$  in the other field.

It is easy to check that

$$\phi = A[\text{dn}^2(\beta x, m) + y], \quad \psi = Bm\text{sn}(\beta x, m)\text{cn}(\beta x, m), \quad (162)$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad (2y + 2 - m)b_1 A^2 = -6\beta^2, \quad (163)$$

$$a_1 = [4(2 - m) + 6y]\beta^2 - [y^2 - (1 - m)]b_1 A^2, \quad a_2 = (2 - m)\beta^2 - [y^2 - (1 - m)]b_1 A^2, \quad (164)$$

and  $y$  is given by Eq. (98).

There is a related PT-invariant solution with PT-eigenvalue  $-1$  in one field and  $+1$  in the other field. In particular,

$$\begin{aligned} \phi &= A[\text{dn}^2(\beta x, m) + y] + iDm\text{cn}(\beta x, m)\text{sn}(\beta x, m), \\ \psi &= Bm\text{cn}(\beta x, m)\text{sn}(\beta x, m) + iF[\text{dn}^2(\beta x, m) + z], \end{aligned} \quad (165)$$

is an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad F = \mp B, \quad y \neq z, \quad (y - z)b_1 A^2 y = -(3/2)\beta^2, \quad (166)$$

$$a_1 = [2 - m + (3/2)(3y + z)]\beta^2, \quad a_2 = [2 - m + (3/2)(3z + y)]\beta^2, \quad (167)$$

and both  $y$  and  $z$  are different; they are given by the two roots of Eq. (100).

There is another PT-invariant solution with PT-eigenvalue  $-1$  in one field and  $+1$  in the other field which is related to the solution (162). In particular,

$$\begin{aligned} \phi &= A[\text{dn}^2(\beta x, m) + y] + iD\sqrt{m}\text{dn}(\beta x, m)\text{sn}(\beta x, m), \\ \psi &= Bm\text{cn}(\beta x, m)\text{sn}(\beta x, m) + iF\sqrt{m}\text{cn}(\beta x, m)\text{dn}(\beta x, m), \end{aligned} \quad (168)$$

is an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad D = \pm A, \quad F = \mp B, \quad (y + 1 - m)b_1 A^2 = -(3/2)\beta^2, \quad (169)$$

$$a_1 = [5 - 4m + (9/2)y]\beta^2 + (1 - m)(y + 1)b_1 A^2, \quad a_2 = [2 - m + (3/2)y]\beta^2 + (1 - m)(y + 1)b_1 A^2, \quad (170)$$

and  $y$  is given by

$$y = \frac{-(5 - 4m) \pm \sqrt{1 - 16m + 16m^2}}{6}. \quad (171)$$

In the limit  $m = 1$ , both the solutions (165) and (168) go over to the corresponding hyperbolic PT-invariant solution

$$\begin{aligned} \phi &= A[\operatorname{sech}^2(\beta x) + y] + iD\operatorname{sech}(\beta x)\tanh(\beta x), \\ \psi &= B\operatorname{sech}(\beta x)\tanh(\beta x) + iF\operatorname{sech}^2(\beta x), \end{aligned} \quad (172)$$

provided

$$b_1 = b_2 = \alpha, \quad D = \pm A, \quad F = \mp B, \quad y = -1/3, \quad z = 0, \quad b_1 A^2 = (9/2)\beta^2, \quad (173)$$

$$a_1 = -(1/2)\beta^2, \quad a_2 = (1/2)\beta^2. \quad (174)$$

It is easy to check that

$$\phi = A[\operatorname{dn}^2(\beta x, m) + y], \quad \psi = B\sqrt{m}\operatorname{sn}(\beta x, m)\operatorname{dn}(\beta x, m), \quad (175)$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad (2y + 1)b_1 A^2 = -6\beta^2, \quad (176)$$

$$a_1 = [4(2 - m) + 6y]\beta^2 - b_1 A^2 y^2, \quad a_2 = (5 - 4m)\beta^2 - b_1 A^2 y^2, \quad (177)$$

while  $y$  is as given by Eq. (98).

There is a related PT-invariant solution with PT-eigenvalue  $-1$  in one field and  $+1$  in the other field.

In particular,

$$\begin{aligned} \phi &= A[\operatorname{dn}^2(\beta x, m) + y] + iD\sqrt{m}\operatorname{dn}(\beta x, m)\operatorname{sn}(\beta x, m), \\ \psi &= B\sqrt{m}\operatorname{dn}(\beta x, m)\operatorname{sn}(\beta x, m) + iF[\operatorname{dn}^2(\beta x, m) + z], \end{aligned} \quad (178)$$

is an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad F = \mp B, \quad y \neq z, \quad (y - z)b_1 A^2 y = -(3/2)\beta^2, \quad (179)$$

$$a_1 = [5 - 4m + (3/2)(3y + z)]\beta^2, \quad a_2 = [5 - 4m + (3/2)(3z + y)]\beta^2, \quad (180)$$

and both  $y$  and  $z$  are different; they are given by the two roots of Eq. (171).

There is another PT-invariant solution with PT-eigenvalue  $-1$  in one field and  $+1$  in the other field which is related to solution (175). In particular,

$$\begin{aligned} \phi &= A[\text{dn}^2(\beta x, m) + y] + iDm\text{cn}(\beta x, m)\text{sn}(\beta x, m), \\ \psi &= B\sqrt{m}\text{dn}(\beta x, m)\text{sn}(\beta x, m) + iF\sqrt{m}\text{cn}(\beta x, m)\text{dn}(\beta x, m), \end{aligned} \quad (181)$$

is an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad F = \mp B, \quad b_1 A^2 y = -(3/2)\beta^2, \quad (182)$$

$$a_1 = [2 - m + (9/2)y]\beta^2 - (1 - m)b_1 A^2, \quad a_2 = [2 - m + (3/2)y]\beta^2 - (1 - m)b_1 A^2, \quad (183)$$

and  $y$  is given by Eq. (100).

In the limit  $m = 1$ , both the solutions (178) and (181) go over to the hyperbolic PT-invariant solution (172).

It is easy to check that

$$\phi = A\sqrt{m}\text{cn}(\beta x, m)\text{dn}(\beta x, m), \quad \psi = B\sqrt{m}\text{sn}(\beta x, m)\text{dn}(\beta x, m), \quad (184)$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad mb_1 A^2 = -6\beta^2, \quad a_1 = (5 - m)\beta^2, \quad a_2 = (5 - 4m)\beta^2. \quad (185)$$

There is a related PT-invariant solution with PT-eigenvalue  $-1$  in one field and  $+1$  in the other field. In particular,

$$\begin{aligned} \phi &= A\sqrt{m}\text{cn}(\beta x, m)\text{dn}(\beta x, m) + iDm\text{cn}(\beta x, m)\text{sn}(\beta x, m), \\ \psi &= B\sqrt{m}\text{dn}(\beta x, m)\text{sn}(\beta x, m) + iF[\text{dn}^2(\beta x, m) + y], \end{aligned} \quad (186)$$

is an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad F = \mp B, \quad y \neq z, \quad (y + 1 - m)b_1 A^2 = (3/2)\beta^2, \quad (187)$$

$$a_1 = (2 - m)\beta^2 + [y^2 - (1 - m)]b_1 A^2, \quad a_2 = (5 - 4m + 3y)\beta^2 + [y^2 - (1 - m)]b_1 A^2, \quad (188)$$

while  $y$  is given by Eq. (171).

It is easy to check that

$$\phi = A\sqrt{m}\text{cn}(\beta x, m)\text{dn}(\beta x, m), \quad \psi = Bm\text{sn}(\beta x, m)\text{cn}(\beta x, m), \quad (189)$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad b_1 A^2 = -6\beta^2, \quad a_1 = (5m - 1)\beta^2, \quad a_2 = (5m - 4)\beta^2. \quad (190)$$

There is a related PT-invariant solution with PT-eigenvalue  $-1$  in one field and  $+1$  in the other field.

In particular,

$$\begin{aligned} \phi &= A\sqrt{m}\text{cn}(\beta x, m)\text{dn}(\beta x, m) + iD\sqrt{m}\text{dn}(\beta x, m)\text{sn}(\beta x, m), \\ \psi &= Bm\text{cn}(\beta x, m)\text{sn}(\beta x, m) + iF[\text{dn}^2(\beta x, m) + y], \end{aligned} \quad (191)$$

is an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad F = \mp B, \quad y \neq z, \quad yb_1 A^2 = (3/2)\beta^2, \quad (192)$$

$$a_1 = [(2 - m) + (3/2)y]\beta^2 + (1 - m)b_1 A^2, \quad a_2 = [2 - m + (9/2)]\beta^2 + (1 - m)b_1 A^2, \quad (193)$$

while  $y$  is given by Eq. (100).

In the limit  $m = 1$ , both the solutions (186) and (191) go over to the hyperbolic PT-invariant solution (172).

It is easy to check that

$$\phi = A \left[ \frac{(1 - m)}{\text{dn}^2(\beta x, m)} + y \right], \quad \psi = Bm \frac{\text{sn}(\beta x, m)\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)}, \quad (194)$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$\begin{aligned} b_1 &= b_2 = \alpha, \quad B = \pm A, \quad (2y + 2 - m)b_1 A^2 = -6\beta^2, \\ a_1 &= [4(2 - m) + 6y]\beta^2 - [y^2 - (1 - m)]b_1 A^2, \quad a_2 = (2 - m)\beta^2 - [y^2 - (1 - m)]b_1 A^2, \end{aligned} \quad (195)$$

while  $y$  is given by Eq. (98).

The PT-invariant combination

$$\begin{aligned}\phi &= A \left[ \frac{(1-m)}{\text{dn}^2(\beta x, m)} + y \right] + iDm \frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} \\ \psi &= Bm \frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} + iF \left[ \frac{(1-m)}{\text{dn}^2(\beta x, m)} + z \right]\end{aligned}\quad (196)$$

is also an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad D = \pm A, \quad F = \mp B, \quad 2b_1 A^2 (y - z) = -3\beta^2, \quad (197)$$

$$a_1 = [2 - m + (3/2)(3y + z)]\beta^2, \quad a_2 = [2 - m + (9/2)(y + 3z)]\beta^2, \quad (198)$$

and  $y$  and  $z$  are unequal; they are given by the two roots of Eq. (100).

### Case III: Solutions With PT Eigenvalue +1 in Both The Fields

Finally, let us discuss PT-invariant solutions in terms of Lamé polynomials of order two with PT-eigenvalue +1 in both the fields. In this context we first note that

$$\phi = A \left[ \frac{(1-m)}{\text{dn}^2(\beta x, m)} + y \right], \quad \psi = B\sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)}, \quad (199)$$

is an exact solution to the coupled Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad B = \pm A, \quad (2y+1)b_1 A^2 = -6\beta^2, \quad a_1 = [4(2-m)+6y]\beta^2 - b_1 A^2 y^2, \quad a_2 = (5-4m)\beta^2 - b_1 A^2 y^2, \quad (200)$$

while  $y$  is given by Eq. (98).

The PT-invariant combination

$$\begin{aligned}\phi &= A \left[ \frac{(1-m)}{\text{dn}^2(\beta x, m)} + y \right] + iDm \frac{\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)} \\ \psi &= B\sqrt{m} \frac{\text{cn}(\beta x, m)}{\text{dn}^2(\beta x, m)} + iF\sqrt{m(1-m)} \frac{\text{sn}(\beta x, m)}{\text{dn}^2(\beta x, m)}\end{aligned}\quad (201)$$

is also an exact solution to Eqs. (119), (120) provided

$$b_1 = b_2 = \alpha, \quad D = \pm A, \quad F = \mp B, \quad 2b_1 A^2 = -3\beta^2, \quad (202)$$

$$a_1 = [2 - m + (9/2)y]\beta^2 - (1-m)b_1 A^2, \quad a_2 = [2 - m + (3/2)y]\beta^2 - (1-m)b_1 A^2, \quad (203)$$

while  $y$  is given by Eq. (100).

## 4.2 Coupled KdV Equations

We now consider a coupled KdV model [31] which has received some attention in the literature. In our recent paper [12], we have already obtained two PT-invariant solutions with PT-eigenvalue +1 in both the fields. We now show that the same model has another novel PT-invariant periodic solution with PT-eigenvalue +1 in both the fields.

The coupled KdV equations are

$$\begin{aligned} u_t + \alpha u u_x + \eta v v_x + u_{xxx} &= 0, \\ v_t + \delta u v_x + v_{xxx} &= 0. \end{aligned} \quad (204)$$

It is easy to check that the coupled Eqs. (204) have the periodic solution

$$u = \frac{A(1-m)}{\text{dn}^2[\beta(x-ct), m]}, \quad v = \frac{B(1-m)}{\text{dn}^2[\beta(x-ct), m]}, \quad (205)$$

provided

$$\delta A = 12\beta^2, \quad \eta B^2 = (\delta - \alpha)A^2, \quad c = 4(2-m)\beta^2. \quad (206)$$

Remarkably, the same model also admits the PT-invariant periodic solution

$$\begin{aligned} u &= \frac{A(1-m)}{\text{dn}^2[\beta(x-ct), m]} + iDm\sqrt{1-m} \frac{\text{sn}[\beta(x-ct), m]\text{cn}[\beta(x-ct), m]}{\text{dn}^2[\beta(x-ct), m]}, \\ v &= \frac{B(1-m)}{\text{dn}^2[\beta(x-ct), m]} + iFm\sqrt{1-m} \frac{\text{sn}[\beta(x-ct), m]\text{cn}[\beta(x-ct), m]}{\text{dn}^2[\beta(x-ct), m]}, \end{aligned} \quad (207)$$

provided

$$D = \pm A, \quad F = \pm B, \quad \delta A = 6\beta^2, \quad \eta B^2 = (\delta - \alpha)A^2, \quad c = (2-m)\beta^2. \quad (208)$$

Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated. It is worth pointing out that this nonsingular solution vanishes in the hyperbolic limit of  $m = 1$ .

## 4.3 Coupled KdV-mKdV Model

Recently we had considered [12] a coupled KdV-mKdV model [32]

$$\begin{aligned} u_t + u_{xxx} + 6uu_x + 2\alpha uvv_x &= 0, \\ v_t + v_{xxx} + 6v^2v_x + \gamma vu_x &= 0, \end{aligned} \quad (209)$$



and obtained PT-invariant solutions with PT-eigenvalue +1 in both the fields. We now show that the same model also admits PT-invariant solutions with PT-eigenvalue +1 in one field and  $-1$  in the other field.

Let us first note that

$$u = A \operatorname{dn}^2[\beta(x - ct), m] + Ay, \quad v = B \sqrt{m} \operatorname{sn}[\beta(x - ct), m], \quad (210)$$

is an exact solution of the coupled Eqs. (209) provided

$$4\gamma A - 12B^2 = 12\beta^2 = 6A - \alpha B^2, \quad c = -(1 + m)\beta^2, \quad y = -\frac{(3 - m)}{4}. \quad (211)$$

It is easy to check that the same model also admits the PT-invariant solution

$$\begin{aligned} u &= A[\operatorname{dn}^2[\beta(x - ct), m] + y] + iD\sqrt{m} \operatorname{sn}[\beta(x - ct), m] \operatorname{dn}[\beta(x - ct), m], \\ v &= B\sqrt{m} \operatorname{sn}[\beta(x - ct), m] + iF \operatorname{dn}[\beta(x - ct), m], \end{aligned} \quad (212)$$

provided

$$D = \pm A, \quad F = \mp B, \quad 2\gamma A - 12B^2 = 3\beta^2 = 3A - \alpha B^2, \quad c = -\frac{(2m - 1)}{2}\beta^2, \quad y = -\frac{(3 - 2m)}{4}. \quad (213)$$

Note that the signs of  $D = \pm A$  and  $F = \mp B$  are correlated. Further, the same model also admits another PT-invariant solution

$$\begin{aligned} u &= A[\operatorname{dn}^2[\beta(x - ct), m] + y] + iDm \operatorname{sn}[\beta(x - ct), m] \operatorname{cn}[\beta(x - ct), m], \\ v &= B\sqrt{m} \operatorname{sn}[\beta(x - ct), m] + iF\sqrt{m} \operatorname{sn}[\beta(x - ct), m], \end{aligned} \quad (214)$$

provided

$$D = \pm A, \quad F = \mp B, \quad 2\gamma A - 12B^2 = 3\beta^2 = 3A - \alpha B^2, \quad c = -\frac{(2 - m)}{2}\beta^2, \quad G = -\frac{(2 - m)}{4}. \quad (215)$$

Note that the signs of  $D = \pm A$  and  $F = \mp B$  are correlated.

In the limit  $m = 1$ , both the solutions (212) and (214) go over to the hyperbolic mixed PT-invariant solution

$$\begin{aligned} u &= A[\operatorname{sech}^2\beta(x - ct) + y] + iD \operatorname{sech}\beta(x - ct) \tanh\beta(x - ct), \\ v &= B \tanh\beta(x - ct) + iF \operatorname{sech}\beta(x - ct), \end{aligned} \quad (216)$$

provided

$$D = \pm A, \quad F = \mp B, \quad 2\gamma A - 12B^2 = 3\beta^2 = 3A - \alpha B^2, \quad c = -1/2\beta^2, \quad y = -1/4. \quad (217)$$

Note that the signs of  $D = \pm A$  and  $F = \mp B$  are correlated.

Yet another solution to the coupled Eqs. (209) is

$$u = \frac{A(1-m)}{\text{dn}^2[\beta(x-ct), m]} + Ay, \quad v = B\sqrt{m(1-m)} \frac{\text{sn}[\beta(x-ct), m]}{\text{dn}[\beta(x-ct), m]}, \quad (218)$$

provided

$$4\gamma A + 12B^2 = 12\beta^2 = 6A + \alpha B^2, \quad c = (2m-1)\beta^2, \quad y = -\frac{(3-2m)}{4}. \quad (219)$$

It is easy to check that the same model also admits the PT-invariant solution

$$\begin{aligned} u &= \frac{A(1-m)}{[\text{dn}^2[\beta(x-ct), m]} + y + iD\sqrt{m(1-m)} \frac{\text{cn}[\beta(x-ct), m]\text{sn}[\beta(x-ct), m]}{\text{dn}^2[\beta(x-ct), m]}, \\ v &= B\sqrt{m(1-m)} \frac{\text{sn}[\beta(x-ct), m]}{\text{dn}[\beta(x-ct), m]} + iF\sqrt{m} \frac{\text{cn}[\beta(x-ct), m]}{\text{dn}[\beta(x-ct), m]}, \end{aligned} \quad (220)$$

with mixed PT-eigenvalues provided

$$D = \pm A, \quad F = \pm B, \quad 2\gamma A - 12B^2 = 3\beta^2 = 3A - \alpha B^2, \quad c = -\frac{(2-m)}{2}\beta^2, \quad y = -\frac{(2-m)}{4}. \quad (221)$$

Note that the signs of  $D = \pm A$  and  $F = \pm B$  are correlated.

## 5 Summary and Conclusions

In this paper we have in a sense completed the program which we had initiated recently. In particular, in a recent publication [12] we have shown through several examples that whenever a real nonlinear equation admits solutions in terms of  $\text{sech}x$  ( $\text{sech}^2x$ ), then the same equation also admits complex PT-invariant solutions with PT-eigenvalue  $+1$  in terms of  $\text{sech}x \pm i \tanh x$  ( $\text{sech}^2x \pm i \text{sech}x \tanh x$ ). Further, we have also shown that such PT-invariant solutions also exist in the corresponding periodic case. On the other hand, in this paper we have shown through several examples that whenever a real nonlinear equation admits kink solutions in terms of  $\tanh x$ , then the same equation also admits complex PT-invariant kink solutions with PT-eigenvalue  $-1$  in terms of  $\tanh x \pm i \text{sech}x$ . We have also shown that both the kink and the

PT-invariant kink solutions have the same topological charge as well as the same kink energy. In addition, for several kink bearing equations we have explicitly shown that even the PT-invariant kink solution is linearly stable. In the case of  $\phi^4$  kink we have shown that like the usual  $\phi^4$  kink, the PT-invariant  $\phi^4$  kink too has two modes and the corresponding eigenenergies are in fact identical for the usual and the PT-invariant kink. We believe this is quite significant and the PT-invariant kink can have some physical relevance. It would be worthwhile to examine this issue in detail.

Further, we have shown that such PT-invariant solutions also exist in the corresponding periodic case. In particular, we have shown through several examples that whenever a nonlinear equation admits periodic solutions in terms of Jacobi elliptic function  $\text{sn}(x, m)$ , then the same equation will also admit complex PT-invariant periodic solutions with PT-eigenvalue  $-1$  in terms of  $\text{sn}(x, m) \pm i\text{cn}(x, m)$  as well as  $\text{sn}(x, m) \pm i\text{dn}(x, m)$ . Moreover, in a few coupled models we have also shown the existence of PT-invariant periodic solutions in terms of Lamé polynomials of order one and two and with PT-eigenvalue being either  $+1$  or  $-1$  in both the fields or  $+1$  in one field and  $-1$  in the other field.

These results raise several important questions. The obvious open question is whether these result are true in general. It would be nice if one can prove this in general, both in the hyperbolic as well as in the periodic case. In the absence of a general proof, it is worthwhile to look at more examples and see if this observation is true in the additional cases or if there are some exceptions. The other obvious question is: what could be the deeper reason because of which such solutions exist? Another question is about the significance of such solutions for real nonlinear equations. In this context we would like to remark that the symmetry of solutions of a nonlinear equation need not be the same as that of the nonlinear equation but it could be less. We believe that auto-Bäcklund transformations may also provide the solutions considered here in the case of both integrable [33] and non-integrable models [34].

Normally, the complex solutions of a real nonlinear equation are not of relevance. However, being PT-invariant complex solutions, we believe they could have some physical significance including in coupled optical waveguides [1, 8, 13]. One pointer in this direction is the fact that for both the KdV and the mKdV equations, which are integrable equations, we have checked that the first three constants of motion for the PT-invariant complex solutions of both the KdV and the mKdV equations are in fact real but have

different values than those for the usual hyperbolic solution (and we suspect that in fact all the constants of motion would be real and would be different than those for the real hyperbolic solution) thereby suggesting that such solutions could be physically interesting. Thus, it would be worthwhile to study the stability of such PT-invariant solutions, which may shed some light on the possible significance of these solutions. We hope to address some of these issues in the near future.

## 6 Acknowledgments

One of us (AK) is grateful to Indian National Science Academy (INSA) for the award of INSA Senior Scientist position at Savitribai Phule Pune University. This work was supported in part by the U.S. Department of Energy.

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